CONNING®

NAIC Scenario Set Technical Documentation

Interest Rates Model

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1 NAIC Basic Data Set

The Basic Data Set provided free of charge to insurers is the standard scenario file set delivered as part of the NAIC scenario service. Users can access the scenarios online by downloading a file containing stochastic scenarios from the GEMS® Economic Scenario Generator (ESG) for real-world interest rates, equity and bond fund returns. The typical application for these scenarios is in calculations of life and annuity Statutory reserves according the Valuation Manual (e.g., VM-20, VM-21) and capital under the NAIC RBC requirements (e.g., C3 Phase 1, C3 Phase 2).

In this document the technical specification of the underlying stochastic model of the ESG used for producing government bond yields and returns on bond funds for the Basic Data Set are described.

2 Interest Rate Model

Scenarios for the non-defaultable term structure of interest rates (i.e. Treasury yields) as well as bond fund returns are simulated using a multi-factor affine short rate model. Some of the reasons that this model has been selected are:

- Realistic yield curve dynamics with a small number of parameters.
- Arbitrage free dynamics making the model applicable to risk-neutral and real-world applications.
- Closed form bond and semi-closed form derivative price formulas.
- Well understood estimation and calibration procedures.
- Production of parallel shifts, twists or steeping and curvature in simulated yield curves.

2.1 3 Factor Model Specification

The underpinnings of the model are a 3 factor extension to the well know Cox-Ingersoll-Ross (CIR) modeling framework. In this model the dynamics of yield curves are governed by three independent stochastic processes referred to as state variables. These state variables are labeled $X_i(t)$, where i=1,2,3 and t is time, indicating that the value of these variables change through time.

Three factors are chosen because they allow for the modeling of the three predominant types of yield curve movement observed in real market data; parallel shifts, steepening and curvature (sometimes referred to as shift, twist and butterfly/smile).

The 3 state variables $X_i(t)$ are governed by an extended version of the standard CIR stochastic differential equation (SDE);

$$dX_i(t) = (\theta_i + \lambda_i^0 + (\lambda_i^1 - \kappa_i)X_i(t))dt + \sigma_i\sqrt{X_i(t)}dW_i(t)$$

Where,

- κ_i controls the mean reversion speed and level
- θ_i controls the mean reversion level
- λ_i^0 and λ_i^1 are risk premia
- σ_i controls the volatility
- $dW_i(t)$ is the change in the Wiener process.

The formulation of the model implemented in GEMS® also allows for the simulation of negative interest rates, in line with what has been observed in market data in many geographies in the last 20 years.

In the real-world formulation of this model used to produce the Basic Data Set, risk premia are included in the above formula. These risk premia allow the model to better capture the dynamics of real government bond markets as well as defining the change of measure between real-world and risk-neutral. The three-factor formulation of the model can produce kurtose and skewed distributions which match historically observed distributions well. The mean reverting properties of the model produce stable long-term simulations with well constrained tails. This is in contrast to the commonly used log normal models, such as Libor market models (LMM) and Black Karasinski models, which tend to produce unrealistic or explosive interest rates, especially over long simulation horizons, making them less suitable for real-world applications.

2.2 Yield Curve Simulation and Construction

The interest rate model falls into a class of models called affine short rate models. In these models a stochastic short rate (in the 1 factor case) or a set of stochastic state variables (in the multi factor case) are simulated. The price of a zero coupon bond at time t maturing at time τ, P(t,τ) has the form;

$$P(t,\tau) = e^{\sum_{i=1}^{3}(A_i(\tau)+ B_i(\tau)X_i(t))}$$

Where in the case of the model described here, $X_i(t)$ are the three state variables of the interest rate model described in section 2.1, and $A_i(\tau)$ and $B_i(\tau)$ are so call auxiliary functions of the model. The auxiliary functions for the CIR model are given by;

$$A_i(\tau) = \frac{2\theta_i}{\sigma_i^2}\ln\left(\frac{2\gamma_i\, e^{(\gamma_i+\kappa_i)\tau/2}}{(\gamma_i+\kappa_i)(e^{\gamma_i\tau}-1)+2\gamma_i}\right)$$

and;

$$B_i(\tau) = \frac{-2(e^{\gamma_i\tau}-1)}{(\gamma_i+\kappa_i)(e^{\gamma_i\tau}-1)+2\gamma_i}$$

where;

$$\gamma_i = \sqrt{\kappa_i^2 + 2\sigma_i^2}$$

By extension the continuously compounded yield curve can be constructed by rearranging the pricing formula to give;

$$y(t,\tau) = \frac{-1}{\tau}\left(\sum_{i=1}^{3} A_i(\tau) + \sum_{i=1}^{3} B_i(\tau)X_i\,(t)\right)$$

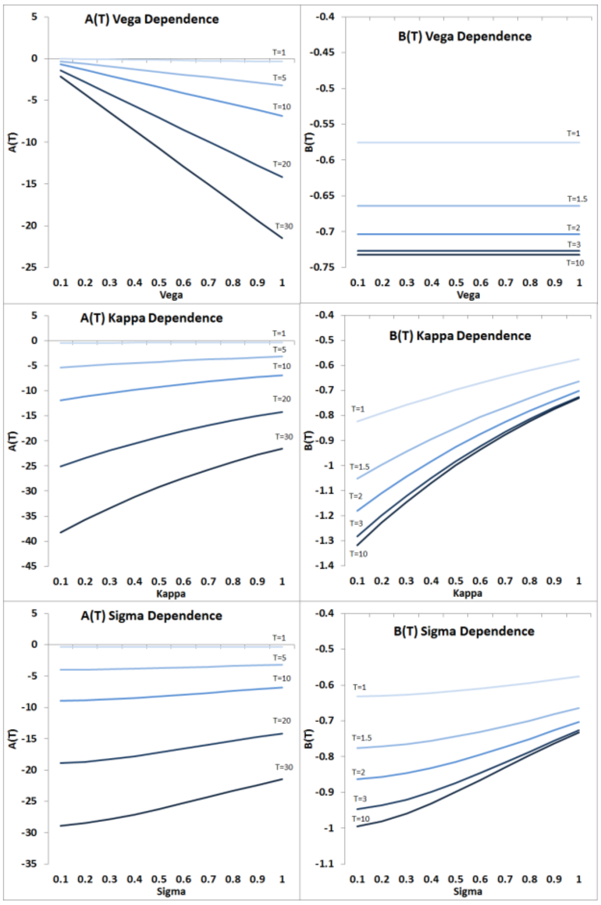

Figure 1. The parameter and maturity dependencies of the auxiliary functions of the Cox Ingersoll Ross interest rate model.

Figure 1 shows the dependencies of the two auxiliary functions A and B on the model parameters, and the bond maturity given fixed values for the other parameters. Observing how the value changes for different points on the yield curve as we change the model parameters gives a visual representation of how different model parameters enable different yield curve shapes to emerge in a multi-factor model with this structure.

Note that the auxiliary functions are only dependent on the risk neutral model parameters (i.e. the parameters λ do not enter the functions) and the time to maturity of the bond under consideration. Therefore, given the simulated state variables at any point in time and the parameters of the model, the yield curve can be constructed from the given pricing formula.

An important aspect of any yield curve model is its ability to produce a wide range of simulated yield curve shapes. Figure 2 shows four selected simulated yield curves from the GEMS® model. This demonstrates the extent to which the model is able to produce quite flat yield curves (e.g. path 20), Steep as well as flatter yield curves with negative rates (e.g. paths 36 and 251) and even inverted yield curves (e.g. path 44). The curves shown also demonstrate different degrees of curvature.

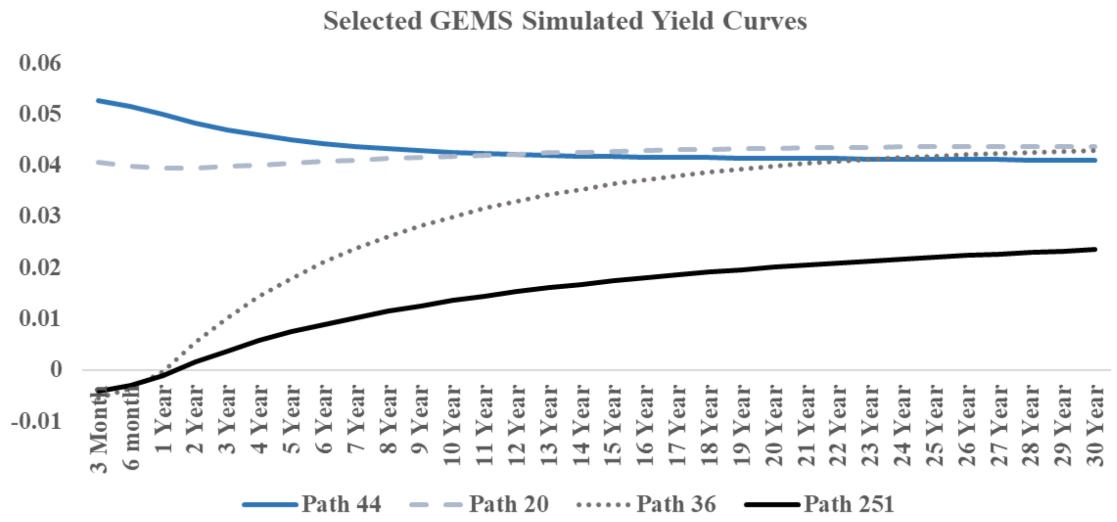

Figure 2 Simulated yield curves from the GEMS® Economic Scenario Generator.

2.3 Initial Yield Curve Fitting

In its standard form as described the multifactor CIR model is only able to approximately fit the initial market yield curve. For most practical applications it is a requirement that the initial yield curve and sometimes yields for extrapolated maturities must be fit to a high degree of accuracy. To enable such fitting the GEMS® model employs a deterministic shift function exogenously from the stochastic differential equation used to drive the state variables. Under this scheme the formula for the initial or t=0 continuously compounded yield of a bond of maturity τ is modified by a shift function l(t) such that;

$$y(t=0,\tau) = \frac{-1}{\tau}\left(\left(-\int_0^\tau l(s)\,ds\right) + \sum_{i=1}^{3} A_i(\tau) + \sum_{i=1}^{3} B_i(\tau) X_i\,(t=0)\right)$$

Given that at time zero the auxiliary functions A and B and the state variables $X_i(t=0)$ are fixed for all paths, the function l(t) can be found which recovers any given market curve. Since the values of the function l(t) can take negative values this mechanism also allows for the model to produce negative simulated yields while maintaining strictly positive state variables $X_i(t)$, which is a requirement under the CIR process due to the square root in the diffusion term.

Figure 3 shows a fit to an initial yield curve for United States Treasuries and demonstrates the extent to which the market yield curve is accurately fit. The model can also fit initial yield curves with significantly negative starting values, including when longer tenors are negative.

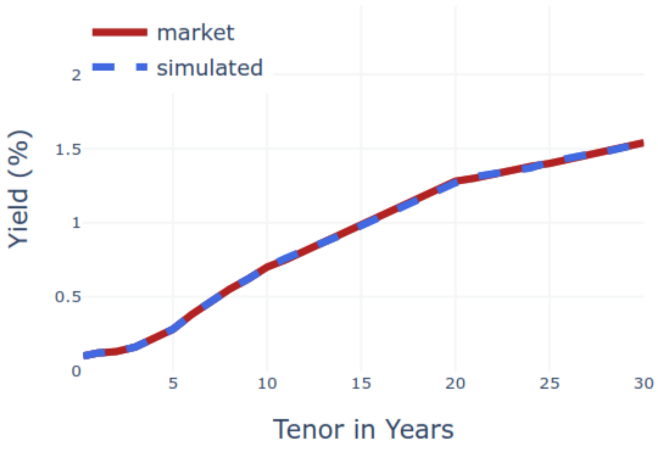

Figure 3 3-Factor model fit to initial yield curve for United States Treasuries.

3 Mean Reversion

The GEMS® three-factor model of interest rates is a mean reverting model. The model has a well-defined mean reversion level for the state variables $X_i(t)$ defined by the parameters θ_i and κ_i, and the speed of mean reversion is controlled through the parameter κ_i.

More explicitly we can see that for a process with a drift term of the form;

$$dX_i(t) = (\theta_i - \kappa_i X_i(t))dt$$

such as the square root process used in the GEMS® three factor model, that the (mean) value of the process at time t=1 will be given by;

$$X_i(t=1) = X_i(t=0) + dX_i(t=1)$$

$$= X_i(t=0) + (\theta_i - \kappa_i X_i(t=0))$$

$$= (1-\kappa_i)X_i(t=0) + \theta_i$$

Following the same logic the value of the process at t=2 will be;

$$X_i(t=2) = (1-\kappa_i)^2 X_i(t=0) + (1-\kappa_i)\theta_i + \theta_i$$

and at t=3

$$X_i(t=3) = (1-\kappa_i)^3 X_i(t=0) + (1-\kappa_i)^2\theta_i + (1-\kappa_i)\theta_i + \theta_i$$

And more generally;

$$X_i(t) = (1-\kappa_i)^t X_i(t=0) + \theta_i \sum_{j=0}^{t-1} (1-\kappa_i)^j$$

And from this we can see that the mean reversion property of the process is that the first summand converges to 0 and the second summand converges to θ_i/κ_i as $t \to \infty$.

The mean reversion parameters of the model are usually estimated either from data or based on a set of calibration criteria for the long term mean reversion level of interest rates.

4 Calibration Criteria

The calibration criteria for the models consists of a set of target values for the distributional properties of nominal interest rates at future time horizons. The precise methodology and final calibration targets are currently under discussion. More information will be added to this section when the details are known.

5 Summary

In this document the technical specification and the properties of the interest rate model used for the production of the NAIC Basic Data Set has been described. The GEMS® three factor model of interest rates described represents a highly parsimonious and highly tractable structure with which to generate stochastic term structures and bond returns. The model is efficient to estimate and produces output which is consistent with the dynamics observed in real market data. Prior to scenario production the model is fit to the initial market yield curve. The statistical properties of the simulated model can also be customized to take account of specified or changing calibration criteria.

6 Additional Reading

Cox, J.C., J.E. Ingersoll and S.A. Ross (1985). ***A Theory of the Term Structure of Interest Rates***. Econometrica 53: 385407.

Brigo, Damiano and Fabio Mercurio (2001b). ***A deterministic shift extension of analytically tractable and time-homogeneous short rate models.*** Finance & Stochastics 5 (3): 369388

7 Appendices

7.1 Appendix I – Relevant Tickers

The following tickers may be relevant as validation benchmarks for the stochastic output of the GEMS® interest rate model. Conning does not supply, distribute or directly derive the models from this data and is supplied here for guidance only.

Description	**Ticker**
Treasury 3 month constant maturity	H15T3M Index
Treasury 6 month constant maturity	H15T6M Index
Treasury 1 year constant maturity	H15T1Y Index
Treasury 2 year constant maturity	H15T2Y Index
Treasury 3 year constant maturity	H15T3Y Index
Treasury 5 year constant maturity	H15T5Y Index
Treasury 7 year constant maturity	H15T7Y Index

Treasury 10 year constant maturity	H15T10Y Index
Treasury 20 year constant maturity	H15T20Y Index
Treasury 30 year constant maturity	H15T30Y Index

7.2 Appendix II – Initial State Variable Calculation

For the stochastic process to initialize at time 0, non-zero positive initial state variables must be assigned to the stochastic processes X_i. This is done by first choosing three "pivot yields" from the starting market yield curve and choosing X_i such that the model matches these three points on the yield curve.

The calculation proceeds by choosing three pivot yields with values (y1, y2, y3) and maturities (τ1, τ2, τ3) and applying an inversion of the three-dimensional form of the CIR process yield formula:

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \left[\begin{pmatrix} B_{1,\tau_1}/-\tau_1 & B_{2,\tau_1}/-\tau_1 & B_{3,\tau_1}/-\tau_1 \\ B_{1,\tau_2}/-\tau_2 & B_{2,\tau_2}/-\tau_2 & B_{3,\tau_2}/-\tau_2 \\ B_{1,\tau_3}/-\tau_3 & B_{2,\tau_3}/-\tau_3 & B_{3,\tau_3}/-\tau_3 \end{pmatrix}\right]^{-1} \left[\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} - \begin{pmatrix} A_1/-\tau_1 \\ A_2/-\tau_2 \\ A_3/-\tau_3 \end{pmatrix} - \begin{pmatrix} l_0 \\ l_0 \\ l_0 \end{pmatrix}\right]$$

where for the avoidance of doubt the auxiliary function $B_{1,\tau 1}$ means the function B calculated using the parameters of process X_1 and the maturity τ_1, the auxiliary function $B_{1,\tau 2}$ means the function B calculated using the parameters of process X_1 and the maturity τ_2 etc.
The elements of the vector containing (A_1, A_2, A_3) are calculated as;

$$A_j = \sum_{i=1}^{3} A_{i,\tau_j}$$

meaning that A_1 is the sum of the auxiliary function A using maturity τ_1 and the parameters of the process X_1,X_2 and X_3 sequentially.

If a set of pivot yields is found to yield negative initial state variables which are inadmissible by the CIR process a search is performed for a combination which gives positive values.

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